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CR-LIGHTLIKE WARPED PRODUCT IN GOLDEN SEMI-RIEMANNIAN MANIFOLDS

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Abstract: In this research, we examine CR-lightlike submanifolds $(CRL)sub_M$ of a Golden semi-Riemannian manifold $(SR)_M$. We obtain some interesting theorems on CR-lightlike warped product (WP). We also construct an example of CR-lightlike submanifolds.

Keywords and Phrases: Golden structure, golden semi-Riemannian manifold, CR-lightlike submanifolds, totally umbilical, warped product.

2020 Mathematics Subject Classification: 53C12, 53C15, 53C22, 53C40, 53C56.

1. Introduction

The theory of CR-submanifolds of a Kaehler manifold $(\overline{Q}, q)$ was initiated by Bejancu [5], [6], as a result of generalization of holomorphic (invariant) and totally real (anti-invariant) submanifolds of Kaehler manifolds, where q is a non-degenerate metric tensor. The classical theory of Riemannian submanifolds breaks down if the induced metric tensor is degenerate because the tangent bundle and the submanifold's normal bundle are not complementary i.e. there is an intersection that is not zero. In order to solve this issue, K. L. Duggal and A. Bejancu [12] presented a few novel techniques and explored lightlike submanifolds (also see [1], [12], [15], [20]). Afterwards, research on generalized $(CRL)sub_M$s was initiated and studied in [13], [14], [22]. In [11], Duggal and Bejancu studied the lightlike CR-hypersurfaces of indefinite Kaehler manifolds. Then, totally umbilical CR-submanifolds of Kaehler manifolds and totally umbilical $(CRL)sub_M$s of indefinite Kaehler manifolds were studied by Bejancu [7] and R. Kumar, M. Gogna, R. K. Nagaich [19]. In [21], R. Kumar, J. Kaur, and R. K. Nagaich have researched CR-lightlike product submanifolds, a unique class of $(CRL)sub_M$s.

Using the golden ratio ϕ, which is the real positive root of the structure polynomial equation $f(x) = x^2 - x - 1 = 0$ (thus, $\phi = \frac{(1+\sqrt{5})}{2}$), C. E. Hretcanu and M. Crasmareanu recently introduced and studied golden structures and their geometric properties on Riemannian manifolds [10]. Further, M. Ahmad and M. A. Qayyoom [2], as well as C. E. Hretcanu [17], explored submanifolds in Riemannian manifolds with golden structure. Later on, the lightlike submanifolds of a Golden $(SR)_M$ was studied by N. Poyraz and E. Yasar in [23]. Shankar at.al [25] studied screen semi-invariant ightlike submanifolds of a Golden semi-Riemannian manifold.

Many mathematicians and physicists have researched the concept of warped product manifolds, which was first introduced by Bishop and O'Neill in [8]. Riemannian product manifolds are generalized into these manifolds. For instance, a surface of revolution is a warped product manifold. Warped product CR- submanifolds were introduced by Chen in [9]. He demonstrated that the only warped product CR-submanifold of the type $Q = Q_1 \times_\lambda Q_2$ that exists is the CR-product, in which Q_1 is a fully real submanifold and Q_2 is a holomorphic submanifold of $\overline{Q}$. As a result, he defined a warped product CR-submanifold of type $Q = Q_1 \times_\lambda Q_2$ as a CR-warped product. Later on, warped product and CR warped product submanifolds were explored in many studies (see [3], [4], [16], [18], [24], [26]).

In this paper, we examine some characteristics of CR-lightlike warped product in Golden $(SR)_M$s. We also study some theorems on totally umbilical $(CRL)sub_M$s of Golden $(SR)_M$s.

Abbreviation:
We have used following abbreviations in this paper-
(1) $(SR)_M$: Semi-Riemannian manifold
(2) $(CRL)sub_M$: CR-lightlike submanifolds
(3) (WP) : Warped product

2. Definition and Preliminaries

2.1. Golden Semi-Riemannian Manifold

Definition 2.1. *Consider an $n-$dimensional manifold $(\overline{Q}, q)$ having a $(1, 1)$ tensor field ψ such that*

$$\psi^2 = \psi + I, \tag{2.1}$$

where I is the identity transformation on tangent bundle $\Gamma(\tau\overline{Q})$. Then, the structure ψ is called a golden structure. If ψ is a self-adjoint operator with respect to metric tensor q which is a symmetric non-degenerate $(0,2)$ tensor field on $\overline{Q}$, i.e.

$$q(\psi F, G) = q(F, \psi G) \tag{2.2}$$

for all $F, G \in \Gamma(\tau\overline{Q})$, then the metric q is said to be ψ-compatible and $(\overline{Q}, q, \psi)$ is called Golden $(SR)_M$. From (2.1) and (2.2) we have

$$q(\psi F, \psi G) = q(F, \psi G) + q(F, G)$$

for any $F, G \in \Gamma(\tau\overline{Q})$.

Definition 2.2. *Consider a Golden $(SR)_M$ $(\overline{Q}, \overline{\psi}, \overline{q})$ and the Levi-Civita connection $\overline{\nabla}$ on $\overline{Q}$ with respect to $\overline{q}$. Then $\overline{Q}$ is called indefinite if $\overline{\psi}$ is parallel with respect to $\overline{\nabla}$, that is,*

$$(\overline{\nabla}_F \overline{\psi})G = 0, \forall\, F, G \in \Gamma(\tau\overline{Q}).$$

2.2. Lightlike Submanifolds

Consider a real $(a+d)$-dimentional $(SR)_M$ $(\overline{Q}, \overline{q})$ of index k where $a, d \geq 1, 1 \leq k \leq a+d-1$ and a-dimensional submanifold (Q, q) of $(\overline{Q}, \overline{q})$ where q is the induced metric of $\overline{q}$ on Q. (Q, q) is called a lightlike submanifold of $(\overline{Q}, \overline{q})$ if $\overline{q}$ is degenerate on the tangent bundle τQ of Q, for details see [12]. $\tau Q^{\perp}$ is a subspace of $\tau_x\overline{Q}$ of dimension d which is degenerate such that

$$\tau Q^{\perp} = \cup\{\mu \in \tau_x\overline{Q} : \overline{q}(\mu, \nu) = 0, \forall\, \nu \in \tau_x Q, x \in Q\}.$$

Here, both $\tau_x Q$ and $\tau_x Q^{\perp}$ are degenerate orthogonal subspace but not complementary any more. Consequently, there is a subspace

$$Rad\tau_x Q = \tau_x Q \cap \tau_x Q^{\perp}.$$

We call it radical or null subspace. The submanifold Q is called r-lightlike submanifold if $Rad\tau Q : x \in Q \to Rad\tau_x Q$ defines a radical distribution on Q of rank $r > 0$.

Consider a semi-Riemannian complementary distribution $S(\tau Q)$ known as the screen distribution of $Rad\tau Q$ in τQ. Here we have

$$\tau Q = S(\tau Q) \perp Rad\tau Q,$$

and $S(\tau Q^{\perp})$ is a vector subbundle that is complimentary to $Rad\tau Q$ in $\tau Q^{\perp}$. Consider two vector bundles $tr(\tau Q)$ and $ltr(\tau Q)$ that are complementary (but not orthogonal) on τQ in $\tau\overline{Q}$ and $Rad\tau Q$ in $S(\tau Q^{\perp})^{\perp}$, respectively. We get

$$tr(\tau Q) = S(\tau Q^{\perp}) \perp ltr(\tau Q), \tag{2.3}$$

$$\tau\overline{Q} = \tau Q \oplus tr(\tau Q) = \{Rad\tau Q \oplus ltr(\tau Q)\} \perp S(\tau Q) \perp S(\tau Q^{\perp}). \tag{2.4}$$

Theorem 2.3. [12] *Consider a lightlike submanifold $(Q, q, S(\tau Q))$ of a $(SR)_M$ $(\overline{Q}, \overline{q})$. Then, in $S(\tau Q^{\perp})^{\perp}$ there exists a complementary vector subbundle $ltr(\tau Q)$ of $Rad\tau Q$ and a basis of $\Gamma(ltr(\tau Q)) \mid u$ comprising a smooth section $\{W_i\}$, where u is a coordinate neighbourhood of Q, such that*

$$\overline{q}(W_i, e_i) = 1, \overline{q}(W_i, W_j) = 0,$$

where $\{e_i\}$ is a lightlike basis of $\Gamma(Rad\tau Q)$.

Consider the Levi-Civita connection $\overline{\nabla}$ on $\overline{Q}$. Then, from the decomposition of $\tau\overline{Q}$, the Gauss and Weingarten equations are given by

$$\overline{\nabla}_F G = \nabla_F G + \bar{b}(F, G), \forall\ F, G \in \Gamma(\tau Q), \tag{2.5}$$

$$\overline{\nabla}_F \eta = -A_\eta F + \nabla^{\perp}_F \eta, \ \ \forall\ F \in \Gamma(\tau Q),\ \eta \in \Gamma(tr(\tau Q)), \tag{2.6}$$

where

$$\{\nabla_F G, A_\eta F\} \in \Gamma(\tau Q)$$

and

$$\{\bar{b}(F, G), \nabla^{\perp}_F \eta\} \in \Gamma(tr(\tau Q)).$$

Here, ∇ is a linear connection on Q that is torsion-free, $\bar{b}$ is second fundamental form and A_η is a shape operator.

Let us consider the projection morphism of $tr(\tau Q)$ on $ltr(\tau Q)$ be L and of $tr(\tau Q)$ on $S(\tau Q^{\perp})$ be S. Then, the Gauss and Weingarten equations give

$$\overline{\nabla}_F G = \nabla_F G + \bar{b}^l(F, G) + \bar{b}^s(F, G),$$

$$\overline{\nabla}_F \eta = -A_\eta F + E^l_F \eta + E^s_F \eta,$$

where we put

$$\bar{b}^l(F, G) = L(\bar{b}(F, G)),$$

$$\bar{b}^s(F, G) = S(\bar{b}(F, G)),$$

$$E^l_F \eta = L(\nabla^{\perp}_F \eta),$$

$$E^s_F \eta = S(\nabla^{\perp}_F \eta).$$

In light of the fact that $\bar{b}^l$ and $\bar{b}^s$ are respectively, $\Gamma(ltr(\tau Q))$-valued and $\Gamma(S(\tau Q^{\perp}))$-valued, they are referred to as the lightlike second fundamental form and the screen second fundamental form on Q.

Definition 2.4. *The screen distribution $S(\tau Q)$ is said to be totally geodesic if $\bar{b}^s(F,G) = 0$ for any $F, G \in \Gamma(\tau Q)$.*

3. CR-lightlike Submanifolds

Definition 3.1. *Consider a real 2a-dimensional $(SR)_M$ $(\overline{Q}, \overline{\psi}, \overline{q})$ and its submanifold Q of dimension b. Then Q is said to be a $(CRL)sub_M$ if :*
(a) $\psi(Rad\tau Q)$ is distribution on Q such that

$$Rad\tau Q \cap \psi(Rad\tau Q) = 0;$$

(b) there exist vector bundles $S(\tau Q)$, $S(\tau Q^{\perp})$, $ltr(\tau Q)$, E_0 and $E^{'}$ over Q such that

$$S(\tau Q) = \{\psi(Rad\tau Q) \oplus E^{'}\} \perp E_0; \psi(E_0) = E_0; \psi(E^{'}) = L_1 \perp L_2,$$

where $\Gamma(E_0)$ is a non-degenerate distribution on Q, $\Gamma(L_1)$ and $\Gamma(L_2)$ are vector subbundles of $\Gamma(ltr(\tau Q))$ and $\Gamma(S(\tau Q^{\perp}))$ respectively, and let $Q_1 = \psi(L_1)$ and $Q_2 = \psi(L_2)$.

We can decompose τQ as

$$\tau Q = E \oplus E^{'},$$

where

$$E = Rad\tau Q \perp \psi(Rad\tau Q) \perp E_0.$$

Now, let the projections on E and $E^{'}$ be P and O respectively. For any $F \in \Gamma(\tau Q)$, we write

$$F = PF + OF,$$

where $PF \in E$ and $OF \in E^{'}$. Applying ψ to above equation, we get

$$\psi F = fF + wF,$$

where $fF = \psi PF$ and $wF = \psi OF$. f is a $(1,1)$ tensor field and w is $\Gamma(L_1 \perp L_2)$-valued 1-form on Q. Clearly, $F \in \Gamma(E)$ if and only if $wF = 0$. Now, we set

$$\psi K = mK + nK,$$

for any $K \in \Gamma(tr(\tau Q))$, where mK and nK are sections of τQ and $tr(\tau Q)$ respectively.

Lemma 3.2. *Consider a $(CRL)sub_M$ Q of a Golden $(SR)_M$ $\overline{Q}$. Then, $\nabla_F \overline{\psi} F = \overline{\psi} \nabla_F F$ for any $F \in \Gamma(E_0)$.*
Proof. Let $F, G \in \Gamma(E_0)$. By using Gauss equation, we get

$$q(\nabla_F \overline{\psi} F, G) = q(\overline{\nabla}_F \overline{\psi} F - \bar{b}(F, \overline{\psi} F), G)$$

$$= q(\overline{\nabla}_F \overline{\psi} F, G)$$
$$= q(\overline{\psi} \overline{\nabla}_F F, G)$$
$$= q(\overline{\nabla}_F F, \overline{\psi} G)$$
$$= q(\nabla_F F + \bar{b}(F, F), \overline{\psi} G)$$
$$= q(\nabla_F F, \overline{\psi} G)$$
$$= q(\overline{\psi} \nabla_F F, G)$$

that is,

$$q(\nabla_F \overline{\psi} F - \overline{\psi} \nabla_F F, G) = 0$$

then, the result follows from the non-degeneracy of E_0.

Lemma 3.1. *Consider a $(CRL)sub_M$ Q of a Golden $(SR)_M$ $\overline{Q}$. Then,*

$$(\nabla_F f)G = A_{wG} F + B\bar{b}(F, G), \tag{3.1}$$

$$(\nabla^t_F w)G = C\bar{b}(F, G) - \bar{b}(F, fG), \tag{3.2}$$

for any $F, G \in \Gamma(\tau Q)$, where

$$(\nabla_F f)G = \nabla_F fG - f(\nabla_F G),$$

$$(\nabla^t_F w)G = \nabla^t_F wG - w(\nabla_F G).$$

Proof. For any $F, G \in \Gamma(\tau Q)$, from equation (3.1), we have

$$\psi G = fG + wG$$

$$\overline{\nabla}_F(\psi G) = \overline{\nabla}_F(fG) + \overline{\nabla}_F(wG)$$
$$\psi \overline{\nabla}_F G = \overline{\nabla}_F fG + \overline{\nabla}_F wG$$

Using equations (2.5) and (2.6), we get

$$\psi \nabla_F G + \psi \bar{b}(F, G) = \nabla_F fG + \bar{b}(F, fG) - A_{wG} F + \nabla^t_F wG$$

$$f(\nabla_F G) + w(\nabla_F G) + B\bar{b}(F, G) + C\bar{b}(F, G) = \nabla_F fG + \bar{b}(F, fG) - A_{wG} F + \nabla^t_F wG$$

Comparing tangential and normal parts, we get

$$\nabla_F fG - f(\nabla_F G) = A_{wG} F + B\bar{b}(F, G),$$

$$\nabla^t_F wG - w(\nabla_F G) = C\bar{b}(F, G) - \bar{b}(F, fG).$$

Hence, we get the result.

Theorem 3.4. *Consider a $(CRL)sub_M$ Q of a Golden $(SR)_M$ $\overline{Q}$. Then,*
(i) The almost complex distribution E is integrable if and only if

$$\bar{b}(F, \psi G) = \bar{b}(\psi F, G), \forall\ F, G \in \Gamma(E).$$

(ii) The totally real distribution $E^{'}$ is integrable if and only if

$$A_{\psi\zeta}G = A_{\psi G}\zeta, \ \ \forall\ \zeta, G \in \Gamma(E^{'}).$$

Proof. Since, $\overline{Q}$ is golden semi-Riemannian manifold, therefore

$$\psi\overline{\nabla}_F G = \overline{\nabla}_F \psi G$$

Using equation (2.5), we have

$$\psi(\nabla_F G + \bar{b}(F, G)) = \nabla_F \psi G + \bar{b}(F, \psi G)$$

Interchange F and G in the above equation, we get

$$\psi(\nabla_G F + \bar{b}(G, F)) = \nabla_G \psi F + \bar{b}(G, \psi F)$$

As E is integrable, from above equations, we get

$$\bar{b}(F, \psi G) = \bar{b}(\psi F, G), \forall\ F, G \in \Gamma(E).$$

Similarly, using equation (2.6), we have

$$\psi(-A_\zeta \mu + \nabla^t_\mu \zeta) = -A_{\psi\zeta}\mu + \nabla^t_\mu \psi\zeta$$

Interchange ζ and μ in the above equation, we get

$$\psi(-A_\mu \zeta + \nabla^t_\zeta \mu) = -A_{\psi\mu}\zeta + \nabla^t_\zeta \psi\mu$$

As $E^{'}$ is integrable, from above equations, we get

$$A_{\psi\zeta}\mu = A_{\psi\mu}\zeta, \ \ \forall\ \zeta, \mu \in \Gamma(E^{'}).$$

Example 3.5. Consider a Golden $(SR)_M$ $\overline{Q} = (R^8, \overline{q}, \overline{\psi})$ where $\overline{q}$ is of signature $(+,-,-,+,-,-,+,+)$ w.r.t. basis $\{\partial\mu_1, \partial\mu_2, \partial\mu_3, \partial\mu_4, \partial\mu_5, \partial\mu_6, \partial\mu_7, \partial\mu_8\}$.
By setting $\overline{\psi}\{\partial\mu_1, \partial\mu_2, \partial\mu_3, \partial\mu_4, \partial\mu_5, \partial\mu_6, \partial\mu_7, \partial\mu_8\}$
$= \{\phi\partial\mu_1, \phi\partial\mu_2, \overline{\phi}\partial\mu_3, \phi\partial\mu_4, \phi\partial\mu_5, \phi\partial\mu_6, \overline{\phi}\partial\mu_7, \overline{\phi}\partial\mu_8\}$

where $\phi = \frac{(1+\sqrt{5})}{2}$ and $\overline{\phi} = \frac{(1-\sqrt{5})}{2}$ are the roots of $\mu^2 - \mu - 1 = 0$ and ψ is the golden structure such that $\overline{\psi}^2 = \overline{\psi} + I$. Thus, $\overline{Q}$ is a Golden $(SR)_M$.

Consider a submanifold Q of $(R^8_4, \overline{q}, \overline{\psi})$ given by the equations

$$\mu_1 = y_1 - y_2 + \phi y_3, \;\; \mu_2 = -y_1 + y_2 - \phi y_3, \mu_3 = -y_4 - \overline{\phi} y_5,$$

$$\mu_4 = -y_4 + \phi y_5, \;\; \mu_5 = y_6, \;\; \mu_6 = -y_6, \; \mu_7 = -y_7, \;\; \mu_8 = y_7$$

Here the tangent bundle τQ of Q is spanned by $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\}$ where

$$\zeta_1 = \partial\mu_1 - \partial\mu_2,$$

$$\zeta_2 = -\partial\mu_1 + \partial\mu_2,$$

$$\zeta_3 = \phi\partial\mu_1 - \phi\partial\mu_2,$$

$$\zeta_4 = -\partial\mu_3 + \partial\mu_4,$$

$$\zeta_5 = \overline{\phi}\partial\mu_3 + \phi\partial\mu_4,$$

$$\zeta_6 = \partial\mu_5 - \partial\mu_6,$$

$$\zeta_7 = -\partial\mu_7 + \partial\mu_8.$$

We can see

$$\overline{q}(\zeta_1, \zeta_1) = \overline{q}(\zeta_1, \zeta_2) = ...\overline{q}(\zeta_1, \zeta_7) = 0,$$

$$\overline{q}(\zeta_2, \zeta_1) = \overline{q}(\zeta_2, \zeta_2) = ...\overline{q}(\zeta_2, \zeta_7) = 0.$$

So, we have Q as 2-lightlike submanifold with $Rad(\tau Q) = span\{\zeta_1, \zeta_2\}$ and $S(\tau Q) = span\{\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\}$ and $\overline{\psi}(\zeta_1) = \zeta_3$.
Also, $\overline{\psi}(\zeta_4) = \zeta_5 \in \Gamma(S(\tau Q))$ implies that $E_0 = span\{\zeta_4, \zeta_5\}$. Further, $ltr(\tau Q)$ is spanned by $\{\eta_1, \eta_2\}$, where

$$\eta_1 = \frac{1}{2}\{\partial\mu_1 + \partial\mu_2\}, \;\; \eta_2 = \frac{1}{2}\{-\partial\mu_1 - \partial\mu_2\}.$$

Now,
$S(\tau Q^{\perp}) = Span\{\overline{\psi}(\zeta_6) = \{W_1 = \phi\partial\mu_5 - \phi\partial\mu_6\}, \;\; \overline{\psi}(\zeta_6) = \{W_2 = -\overline{\phi}\partial\mu_7 + \overline{\phi}\partial\mu_8\}\}$
and $E' = span\{\overline{\psi}\eta_1, \overline{\psi}\eta_2, W_1, W_2\}$.
Therefore, Q is a $(CRL)sub_M$ of Golden $(SR)_M$ $\overline{Q}$.

3.1. Totally Umbilical CR-lightlike Submanifolds

Definition 3.5. *A lightlike submanifold (Q, q) of a Golden $(SR)_M$ $(\overline{Q}, \overline{q})$ is called* *totally umbilical in Q if there is a smooth transversal vector field $H \in \Gamma(tr(\tau Q))$ on Q, known as transversal curvature vector field of Q, such that $\forall F, G \in \Gamma(\tau Q)$,*

$$\bar{b}(F,G) = H\overline{q}(F,G).$$

It is obvious that Q is a totally umbilical if and only if on each coordinate neighbourhood u, $H^l \in \Gamma(ltr(\tau Q))$ and $H^s \in \Gamma(S(\tau Q^{\perp}))$ are present, such that

$$\bar{b}^l(F,G) = H^l\overline{q}(F,G),$$

$$\bar{b}^s(F,G) = H^s\overline{q}(F,G).$$

Theorem 3.6. *Consider a totally umbilical $(CRL)sub_M$ Q of a Golden $(SR)_M$ for which the screen distribution is totally geodesic. Then $\overline{q}(H^s, \overline{\psi}^2\eta) = 0$ for each $\eta \in L_2$.*

Proof. Let $\eta \in L_2$ and $F \in E_0$, then for a totally umbilical $(CRL)sub_M$ and totally geodesic screen distribution, we have

$$\overline{q}(\overline{\psi\nabla}_F F, \overline{\psi}\eta) = \overline{q}(\overline{\nabla}_F\overline{\psi}F, \overline{\psi}\eta) = \overline{q}(\nabla_F\overline{\psi}F + \bar{b}^s(F, \overline{\psi}F), \overline{\psi}\eta) = \overline{q}(\nabla_F\overline{\psi}F, \overline{\psi}\eta)$$

Using property of Golden $(SR)_M$ in above relation, we get

$$\overline{q}(\overline{\psi\nabla}_F F, \overline{\psi}\eta) = \overline{q}(\nabla_F\overline{\psi}F, \overline{\psi}\eta) = \overline{q}(\overline{\psi}\nabla_F F, \overline{\psi}\eta)$$

$$= \overline{q}(\overline{\psi}\nabla_F F, \eta) + \overline{q}(\nabla_F F, \eta) = 0 \tag{3.3}$$

Also, we have

$$\overline{q}(\overline{\psi\nabla}_F F, \overline{\psi}\eta) = \overline{q}(\overline{\nabla}_F F, \overline{\psi}\eta) + \overline{q}(\overline{\nabla}_F F, \eta)$$
$$= \overline{q}(\nabla_F F + \bar{b}^s(F,F), \overline{\psi}\eta) + \overline{q}(\nabla_F F + \bar{b}^s(F,F), \eta)$$
$$= \overline{q}(\nabla_F F, \overline{\psi}\eta) + \overline{q}(\nabla_F F, \eta) + \overline{q}(\bar{b}^s(F,F), \overline{\psi}\eta) + \overline{q}(\bar{b}^s(F,F), \eta)$$

As Q is totally umbilical i.e. $\bar{b}^s(F,G) = \overline{q}(F,G)H^s$, so we get

$$\overline{q}(\overline{\psi\nabla}_F F, \overline{\psi}\eta) = \overline{q}(\overline{\psi}\nabla_F F, \eta) + \overline{q}(\nabla_F F, \eta) + \overline{q}(\overline{q}(F,F)H^s, \overline{\psi}\eta) + \overline{q}(\overline{q}(F,F)H^s, \eta)$$

$$= \overline{q}(\overline{q}(F,F)H^s, \overline{\psi}\eta) + \overline{q}(\overline{q}(F,F)H^s, \eta)$$
$$= \overline{q}(F,F)\overline{q}(H^s, \overline{\psi}\eta) + \overline{q}(F,F)\overline{q}(H^s, \eta)$$

From (3.3), we have

$$\overline{q}(F,F)\overline{q}(H^s, \overline{\psi}\eta) + \overline{q}(F,F)\overline{q}(H^s, \eta) = 0$$

$$\overline{q}(F,F)\{\overline{q}(H^s,\overline{\psi}\eta)+\overline{q}(H^s,\eta)\}=0$$

Due to non-degeneracy, we obtain

$$\overline{q}(H^s,\overline{\psi}\eta)+\overline{q}(H^s,\eta)=0$$

$$\overline{q}(H^s,\overline{\psi}\eta+\eta)=0$$

$$\overline{q}(H^s,(\overline{\psi}+I)\eta)=0$$

$$\overline{q}(H^s,\overline{\psi}^2\eta)=0$$

The result follows from the non-degeneracy of q.

4. CR-lightlike Warped Product

Definition 4.1. [24] *Consider two Riemannian manifolds (Q_1,q_1) and (Q_2,q_2) and a non-negative differentiable function λ on Q_1. The (WP) of Q_1 and Q_2 is the Riemannian manifold*

$$Q=Q_1\times_\lambda Q_2=(Q_1\times Q_2,q),$$

where $q=q_1+\lambda^2q_2$. Q_1 is called the base of Q and Q_2 the fibre.
For a (WP) manifold $Q_1\times_\lambda Q_2$, we denote by E_1 and E_2 the distributions defined by the vectors tangent to the base and fibres respectively.

Lemma 4.2. *Consider a (WP) manifold $Q=Q_1\times_\lambda Q_2$. If $F,G\in\tau(Q_1)$ and $X,Y\in\tau(Q_2)$, then*

$$\nabla_F G\in\tau(Q_1), \tag{4.1}$$

$$\nabla_F Y=\nabla_Y F=\frac{F\lambda}{\lambda}Y, \tag{4.2}$$

$$\nabla_X Y=-\frac{q(X,Y)}{\lambda}\nabla\lambda. \tag{4.3}$$

Corollary 4.3. *On a (WP) manifold $Q=Q_1\times_\lambda Q_2$ one has*
(i) Q_1 is totally geodesic in Q,
(ii) Q_2 is totally umbilical in Q.

Definition 4.4. *A $(CRL)sub_M$ Q of a Golden $(SR)_M$ $\overline{Q}$ is called a CR lightlike product if both the distributions E and E' define totally geodesic foliations in Q.*

Lemma 4.5. *Consider a totally umbilical $(CRL)sub_M$ Q of a Golden $(SR)_M$ $\overline{Q}$. Then, a totally geodesic foliation in Q is defined by the distribution E'.*

Theorem 4.6. *Consider a totally umbilical $(CRL)sub_M$ Q of a Golden $(SR)_M$ $\overline{Q}$.*

If $Q = Q_1 \times_\lambda Q_2$ be a (WP) $(CRL)sub_M$, then it is a CR lightlike product.
Proof. Consider the second fundamental form $\overline{b}'$ and the shape operator A' of Q_2, then we have

$$q(\overline{b}'(F,G),\zeta) = q(\nabla_F G,\zeta) = -\overline{q}(G,\overline{\nabla}_F\zeta) = -q(G,\nabla_F\zeta).$$

for $F,G \in \Gamma(E)$ and $\zeta \in \Gamma(E')$. Using (4.2), we obtain

$$q(\overline{b}'(F,G),\zeta) = -(\zeta In\lambda)q(F,G). \tag{4.4}$$

Now, consider the second fundamental form $\hat{\overline{b}}$ of Q_2 in $\overline{Q}$, then

$$\hat{\overline{b}}(F,G) = \overline{b}'(F,G) + \overline{b}^s(F,G) + \overline{b}^l(F,G)$$

for any F,G tangent to Q_2, then using (4.4), we obtain

$$q(\hat{\overline{b}}(F,G),\zeta) = q(\overline{b}'(F,G),\zeta) = -(\zeta In\lambda)q(F,G). \tag{4.5}$$

Since, Q_2 is a holomorphic submanifold of $\overline{Q}$, then we get

$$\hat{\overline{b}}(F,\psi G) = \hat{\overline{b}}(\psi F,G) = \psi\hat{\overline{b}}(F,G),$$

and

$$\hat{\overline{b}}(\psi F,\psi G) = \hat{\overline{b}}(\psi^2 F,G) = \hat{\overline{b}}(\psi F,G) + \hat{\overline{b}}(F,G) = \psi\hat{\overline{b}}(F,G) + \hat{\overline{b}}(F,G)$$

$$\hat{\overline{b}}(F,G) = \hat{\overline{b}}(\psi F,\psi G) - \psi\hat{\overline{b}}(F,G)$$

therefore, we get

$$q(\hat{\overline{b}}(F,G),\zeta) = q(\hat{\overline{b}}(\psi F,\psi G),\zeta) - q(\psi\hat{\overline{b}}(F,G),\zeta)$$

$$q(\hat{\overline{b}}(F,G),\zeta) = -(\zeta In\lambda)q(F,G) - q(\psi\hat{\overline{b}}(F,G),\zeta) \tag{4.6}$$

Adding (4.5) and (4.6), we get

$$q(\psi\hat{\overline{b}}(F,G),\zeta) = 0$$

$$q(\hat{\overline{b}}(F,G),\psi\zeta) = 0$$

It follows that there aren't any components of $\overline{b}(F,G)$ in $L_1 \perp L_2$ for any $F,G \in \Gamma(E)$. This shows that a totally geodesic foliation in Q is defined by the distribution

E. Hence, Q is a CR lightlike product.

Lemma 4.7. *Consider a totally umbilical* $(CRL)sub_M$ Q *of a Golden* $(SR)_M$ $\overline{Q}$, *then for a CR-lightlike* (WP) $Q = Q_1 \times_\lambda Q_2$ *in* $\overline{Q}$, *we have*
(i) $\nabla_F \zeta = -A_{w\zeta} F$
(ii) $\overline{q}(\bar{b}(\psi F, \zeta), \psi\zeta_1) = (FIn\lambda)q(\zeta, \psi\zeta_1) + (FIn\lambda)q(\zeta, \zeta_1)$
for any $F \in \Gamma(E)$ *and* $\zeta, \zeta_1 \in \Gamma(Q_2) \subset \Gamma(E')$.
Proof. (i) For Golden $(SR)_M$ $\overline{Q}$, we have $\overline{\nabla}\psi = 0$, then

$$\overline{\nabla}_F \psi\zeta = (\overline{\nabla}_F J)\zeta + \psi(\overline{\nabla}_F \zeta)$$

for $F \in \Gamma(E)$ and $\zeta \in \Gamma(Q_2)$.
As Q is totally umbilical, therefore, we have

$$\psi(\overline{\nabla}_F \zeta) = -A_{w\zeta}F + \nabla^s_F w\zeta$$

Taking inner product with ψG, where $G \in \Gamma(E)$, we obtain

$$q(\psi(\nabla_F \zeta), \psi G) = -q(A_{w\zeta}F, \psi G) + q(\nabla^s_F w\zeta, \psi G)$$

For Golden $(SR)_M$, we get

$$q(\nabla_F \zeta, \psi G) + q(\nabla_F \zeta, G) = -q(A_{w\zeta}F, \psi G)$$

Using (4.2),

$$q(\nabla_F \zeta, \psi G) = -q(A_{w\zeta}F, \psi G)$$

$$q(\nabla_F \zeta + A_{w\zeta}F, \psi G) = 0$$

Using non-degeneracy of q,

$$\nabla_F \zeta + A_{w\zeta}F = 0$$

$$\nabla_F \zeta = -A_{w\zeta}F$$

(ii) For any $F \in \Gamma(E)$ and $\zeta, \zeta_1 \in \Gamma(Q_2) \subset \Gamma(E')$, we get from Gauss equation

$$q(\bar{b}(\psi F, \zeta), \psi\zeta_1) = q(\overline{\nabla}_\zeta \psi F, \psi\zeta_1) = q(\psi\overline{\nabla}_\zeta F, \psi\zeta_1)$$

$$= q(\nabla_\zeta F, \psi\zeta_1) + q(\nabla_\zeta F, \zeta_1)$$

$$= (FIn\lambda)q(\zeta, \psi\zeta_1) + (FIn\lambda)q(\zeta, \zeta_1)$$

Theorem 4.8. *Consider a locally CR lightlike* (WP) Q *of a Golden* $(SR)_M$ $\overline{Q}$, *then*

$$\overline{q}((\nabla^t_F w)G, \psi\zeta) = -G(In\lambda)\{q(F, \psi\zeta) + q(F, \zeta)\}$$

for any $F, G \in \Gamma(\tau Q)$ *and* $\zeta \in \Gamma(E^{'})$.

Proof. From definition 4.4, we know that totally geodesic foliation in Q is defined by the distribution E, where Q is a CR-lightlike (WP) of a Golden $(SR)_M$ $\overline{Q}$. Then, by using (3.3) for $F, G \in \Gamma(E)$, we get

$$\overline{q}((\nabla^t_F w)G, \psi\zeta) = -\overline{q}(\bar{b}(F, fG), \psi\zeta) = -\overline{q}((\overline{\nabla}_F fG - \nabla_F fG), \psi\zeta)$$

$$= -\overline{q}(\overline{\nabla}_F fG, \psi\zeta) + \overline{q}(\nabla_F fG, \psi\zeta)$$

$$= -\overline{q}(\overline{\nabla}_F \psi G, \psi\zeta) + \overline{q}(\nabla_F fG, \psi\zeta)$$

$$= -\overline{q}(\psi\overline{\nabla}_F G, \psi\zeta) + \overline{q}(f\nabla_F G, \psi\zeta)$$

$$= -\{\overline{q}(\overline{\nabla}_F G, \psi\zeta) + \overline{q}(\overline{\nabla}_F G, \zeta)\} + \overline{q}(f\nabla_F G, \psi\zeta)$$

$$= -\overline{q}(\overline{\nabla}_F G, \psi\zeta + \zeta) + \overline{q}(f\nabla_F G, \psi\zeta)$$

$$= -\overline{q}(\nabla_F G + \bar{b}(F, G), \psi\zeta + \zeta) + \overline{q}(f\nabla_F G, \psi\zeta)$$

$$= -\overline{q}(\nabla_F G, \psi\zeta) - \overline{q}(\nabla_F G, \zeta) + \overline{q}(f\nabla_F G, \psi\zeta) = 0$$

Now, let $F \in \Gamma(E^{'}), G \in \Gamma(E)$. By using (4.2) we have,

$$\overline{q}((\nabla^t_F w)G, \psi\zeta) = -\overline{q}(\bar{b}(F, fG), \psi\zeta) = -\overline{q}(\nabla_F G, \psi\zeta) - \overline{q}(\nabla_F G, \zeta)$$

$$= -G(In\lambda)q(F, \psi\zeta) - G(In\lambda)q(F, \zeta)$$

$$= -G(In\lambda)\{q(F, \psi\zeta) + q(F, \zeta)\}$$

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